Socioeconomic Networks with Long-Range Interactions

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Motivation

- Individuals and firms interact to share information and resources, exchange goods and credit, make new friendships or partnerships etc;
- The structure of the network through which interactions take place may impact on the success of the individual or the productivity of the firm;
- The network of interactions among socioeconomic agents plays an important role for the stability and efficiency of socioeconomic systems;
- Theories about how such interaction networks form are thus essential for a deeper understanding of the development and organization of society as a whole.

Two cultures

- Physics:
  - Characterization of the structure of real networks;
  - Dynamic models (probabilistic), capable of reproducing the observed geometrical structures (Poisson, stretched exponential and scale-free);
  - Shortcoming: network growth mechanisms (e.g. preferential attachment) rely on node degree, which may be only one of the factors determining attachment in social networks;
- Economics:
  - Equilibrium networks;
  - Network formation mechanisms based on utility maximization and costs minimization;
  - Aim at identifying, among the set of equilibrium networks, the geometry that optimizes efficiency in the sense of social benefit;
  - Interested in the stability of equilibrium networks under link deletion, addition or rewiring;
  - Shortcoming: symmetries in the payoff functions mean that equilibrium networks are often too simple in their geometry (stars, complete networks, interlinked starts, etc.);

Jackson and Wolinsky Model

- Agents derive benefit both from their nearest neighbours and from faraway nodes;
- Utility of node i:

$$u_i = w_{ii} + \sum_{j \neq i} w_{ij} \delta^{d_{ij}} - \sum_{j \in \mathcal{V}(i)} c_{ij} \tag{1}$$

- Jackson and Wolinsky study pairwise stability when agents can only update one link at a time;
- Bala and Goyal allow agents to rearrange all their connections at once;
- A configuration is accepted if it increases the utility of the agent;
- In Bala and Goyal, the star network is both efficient and stable for a wide range of the parameters when $\delta = 1$. Multiplicity of network architectures for $0 < \delta < 1$ which could be Nash.

Our model

- Simplified version of Jackson and Wolinsky ($w_{ij} = 1$, $w_{ii} = 0$ and $c_{ij} = c$):

$$u_i = \sum_{l=1}^{l_{max}^{(i)}} \sum_{\{k|d_{ik}=l\}} \delta^l - \sum_{j \in \mathcal{V}(i)} c = \sum_{l=1}^{l_{max}^{(i)}} \delta^l z_l^{(i)} - c z_1^{(i)} \tag{2}$$

- Our approach:
  - Accept a (real world?) network topology (Poisson, scale-free): how does average utility rank compared to other networks?
  - Network growing mechanisms: preferential attachment by node utility;

Analytical results for random networks

- Average utility in generic random networks

$$\overline{u}(\delta) = \sum_{l=1}^{\bar{l}} \delta^l z_l \tag{3}$$

- The generating function formalism leads to:

$$z_l = \left[\frac{z_2}{z_1}\right]^{l-1} z_1 \tag{4}$$

- Replacing (4) in (3) yields:

$$\overline{u}(\delta) = \frac{\delta z_1 \left((\delta Z)^{\bar{l}} - 1\right)}{\delta Z - 1} \tag{5}$$

where $Z = z_2 / z_1$ and

$$\bar{l} = \frac{\ln\left[(N-1)(Z-1)/z_1 + 1\right]}{\ln(Z)} \tag{6}$$

Analytical results for random networks

Star and Poisson random networks

- Average utility of a star network:

$$\overline{u}_{*}(\delta) = \delta z_1 \left(1 + \delta \frac{N-2}{2}\right) \tag{7}$$

where $z_1 = 2(N-1)/N$. For N large, $z_1 \simeq 2$ and $\overline{u}_{*}(\delta) \sim N\delta^2$;

- Poisson random networks are characterized by $z_1 = pN$ and $z_2 = z_1^2$, thus:

$$\overline{u}_P(N, \delta, z_1) = \frac{z_1 \delta \left((\delta z_1)^{\ln(N + \frac{1-N}{z_1}) / \ln(z_1)} - 1 \right)}{\delta z_1 - 1} \tag{8}$$

Analytical results for random networks

Scale-free Networks

$$p_k(\gamma, a) = \frac{1}{\zeta(\gamma, 1+a)}(a+k)^{-\gamma} \quad , a \geq 0 \tag{9}$$

$$\bar{u}_{SF}(N, \delta, \gamma, a) = \frac{\delta z_1(\gamma, a)\left((\delta Z(\gamma, a))^{\bar{l}_{SF}(N,\gamma,a)} - 1\right)}{\delta Z(\gamma, a) - 1} \tag{10}$$

$$\bar{l}_{SF}(N, \gamma, a) = \frac{\ln\left(-\frac{(a+2)(N-1)}{z_1(\gamma,a)} + \frac{z_1(\gamma-1,a)\zeta(\gamma-1,a+1)(N-1)}{z_1(\gamma,a)^2\zeta(\gamma,a+1)} + 1\right)}{\ln\left(-a + \frac{z_1(\gamma-1,a)\zeta(\gamma-1,a+1)}{z_1(\gamma,a)\zeta(\gamma,a+1)} - 1\right)}$$

$$z_1(\gamma, a) = \frac{\Phi(1, \gamma-1, a+1) - a\Phi(1, \gamma, a+1)}{\zeta(\gamma, a+1)} \tag{11}$$

$$z_2(\gamma, a) = \frac{\zeta(\gamma-1, a+1)}{\zeta(\gamma, a+1)} z_1(\gamma-1, a) - (a+1) z_1(\gamma, a) \tag{12}$$

Analytical results for random networks

Scale-free Networks: z_1 and z_2

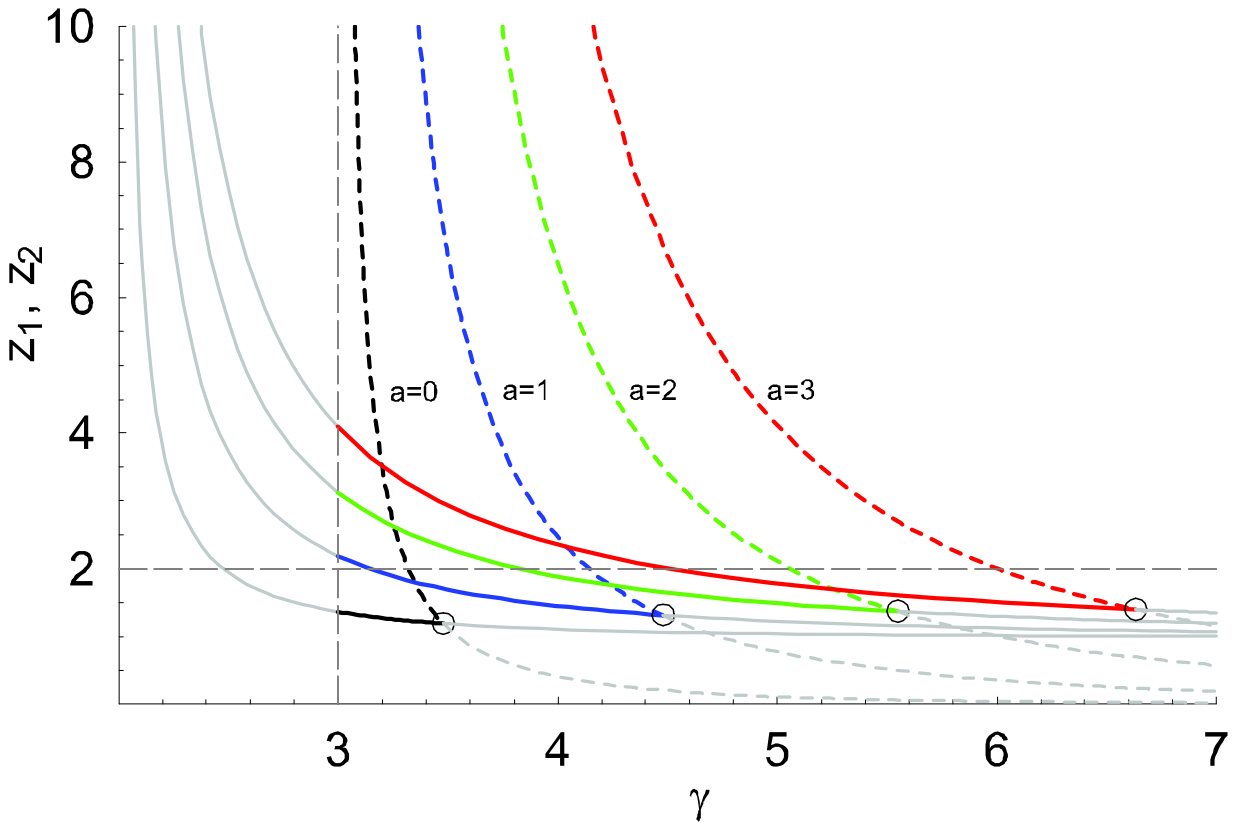


Figure: $z_1(\gamma, a)$ and $z_2(\gamma, a)$ in networks with scale-free degre distribution. We plot $z_1(\gamma, a)$ (full curves) and $z_2(\gamma, a)$ (dashed curves) for $a = 0$ (black), 1 (blue), 2 (green) and 3 (red).

Analytical results for random networks

Average utility in scale-free and Poisson networks

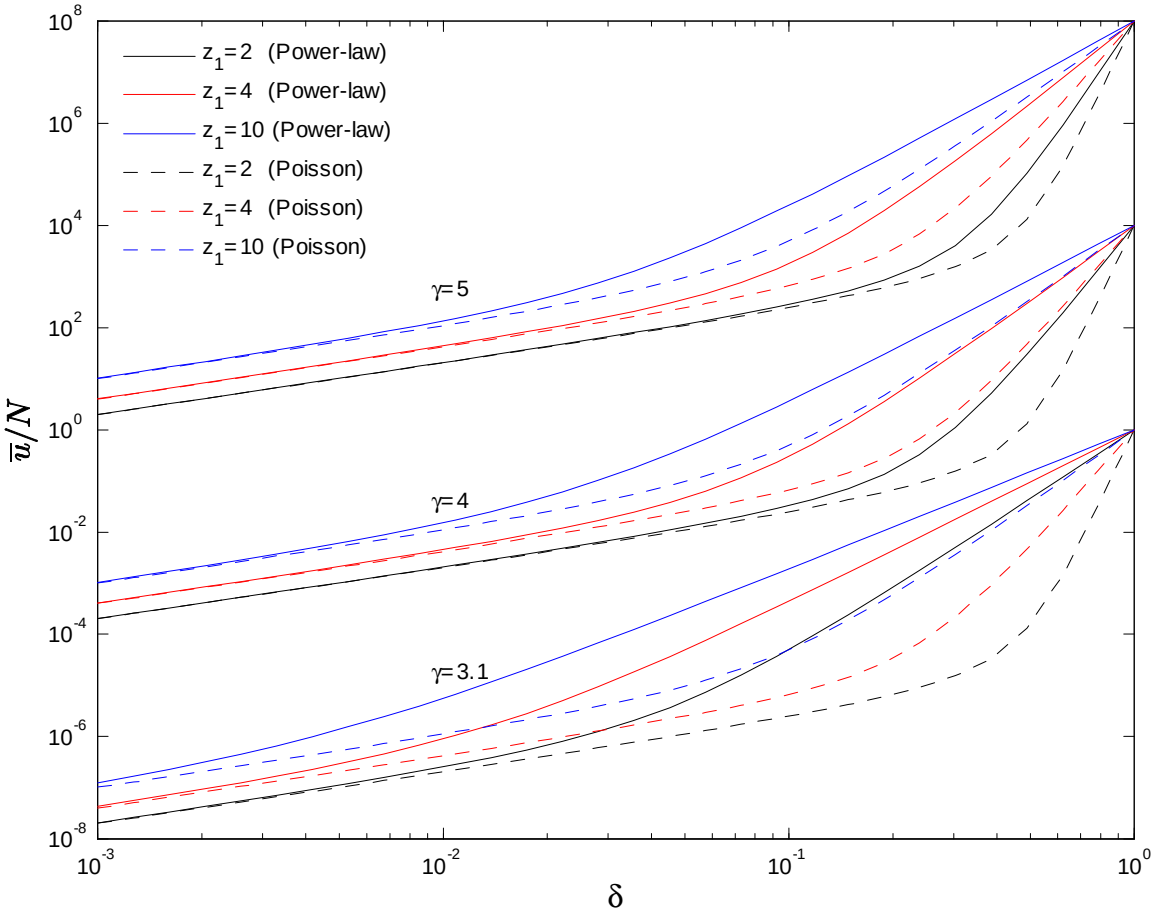


Figure: Scaled average utility in networks with power-law (full curves) and Poisson (dashed curves) degree distributions as a function of δ, z_1 and γ for $N = 10^5$. Curves have been shifted vertically for clarity.

Evolving networks

Preferential attachment

- In the classic Barabási–Albert model, a network is grown by adding, at every time step, a new node that attaches to m existing nodes with a probability proportional to their degree, $\Pi(k_i) = k_i / \sum_{j=1}^{N} k_j$;
- Preferential attachment generates a scale-free probability density of incoming links that leads to the stationary result $p(k) = 2m^2 / k^\gamma$, with $\gamma = 3$ independently of m;
- The linear preferential attachment hypothesis is very sensitive, as the scale-free nature of the network is destroyed by a non-linear attachment rule $\Pi(k_i) \sim k_i^\alpha$;
- There are now several extensions of the preferential attachment mechanism. Of particular relevance to our approach are fitness models:

$$\Pi(k_i) \sim \frac{f_i k_i}{\sum_{j=1}^{N} f_j k_j}. \tag{13}$$

Evolving networks

An extension to the model of Jackson and Wolinksy

- Our contribution: preferential attachment by (time-dependent) node utility: $\Pi_i = \frac{u_i}{\sum_{k=1}^{N} u_k}$;
- All nodes have the same utility for $\delta = 0$ and $\delta = 1$:

$$\begin{cases} u_i = 0 & \forall i \text{ when } \delta = 0 \\ u_i = N & \forall i \text{ when } \delta = 1 \end{cases} \tag{14}$$

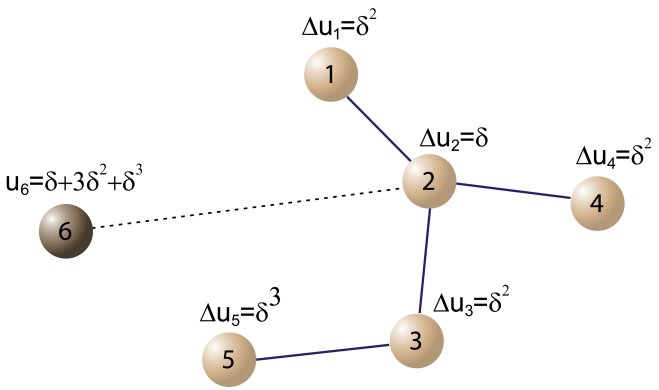
  so attachment happens randomly in these cases and we recover an exponential distribution of node degree;
- The preferential attachment rule is invariant up to multiplicative factors:

$$u_i^{'} = \frac{u_i}{\delta} = k_i + \sum_{l=2}^{l_{max}^{(i)}} \sum_{k \in \mathcal{V}_i^k} \delta^l \tag{15}$$

  where k_i is the degree of node i. Thus, as $\delta \longrightarrow 0$ our model converges to the Barabási-Albert model and the network becomes scale-free.

Evolving networks

Algorithm optimizations



- Nodes i at a higher distance than a certain $l_{\max}$ from new node j receive a contribution $\Delta u_i = \delta^{d(j,i)} < 10^{-precison}$ which is less than the number of significant digits that the computer can store. This maximal distance $l_{\max}$ is defined as

$$10^{-precision} > \delta^{l_{\max}} \Leftrightarrow l_{\max} > -\frac{precision}{\log_{10} \delta} \tag{16}$$

Evolving networks

Average utility

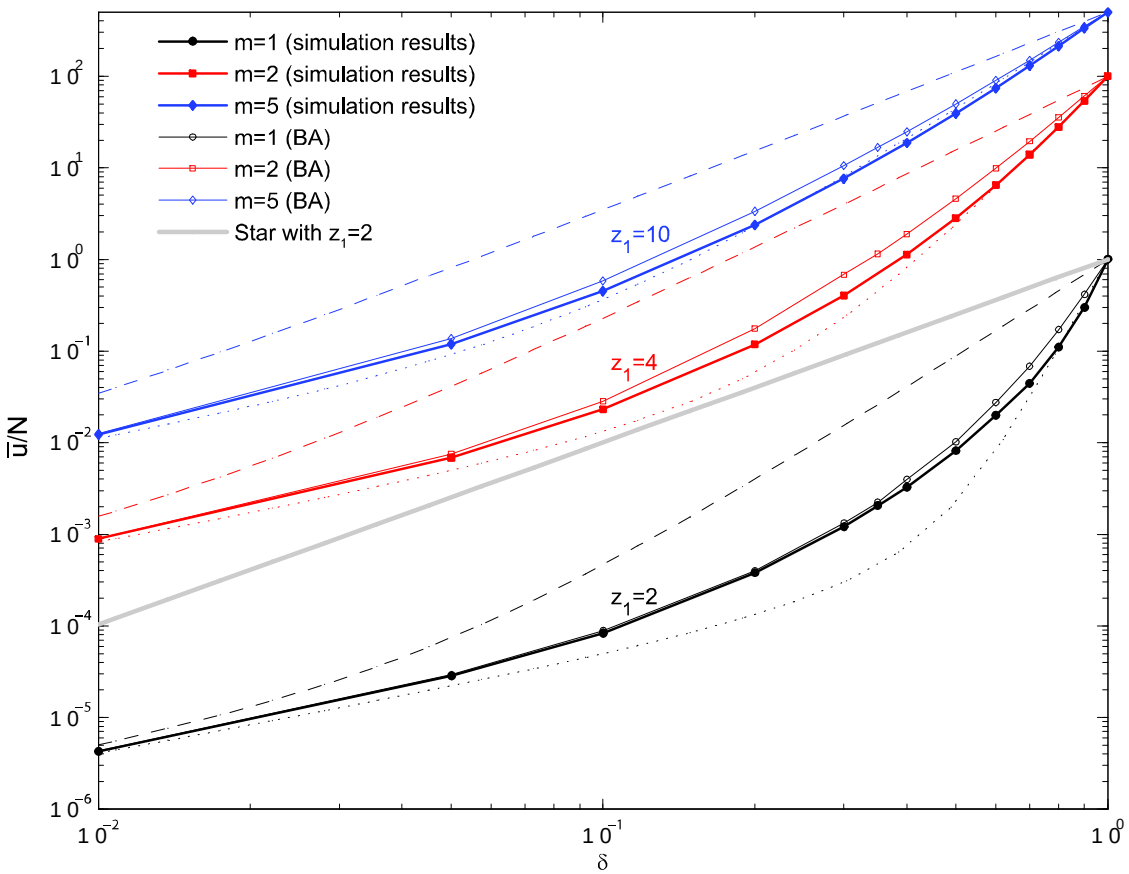


Figure: $\overline{u}/N$ for our model (solid curve and symbols) and the BA model (solid curve, open symbols) for $m = 1$ ($z_1 = 2$), 2 ($z_1 = 4$) and 5 ($z_1 = 10$). Analytical curves for average utility in Poisson (dotted curve) and scale-free (dashed curve) networks for $z_1 = 2, 4$ and 10 and $\gamma = 3.1$ (for scale-free networks).

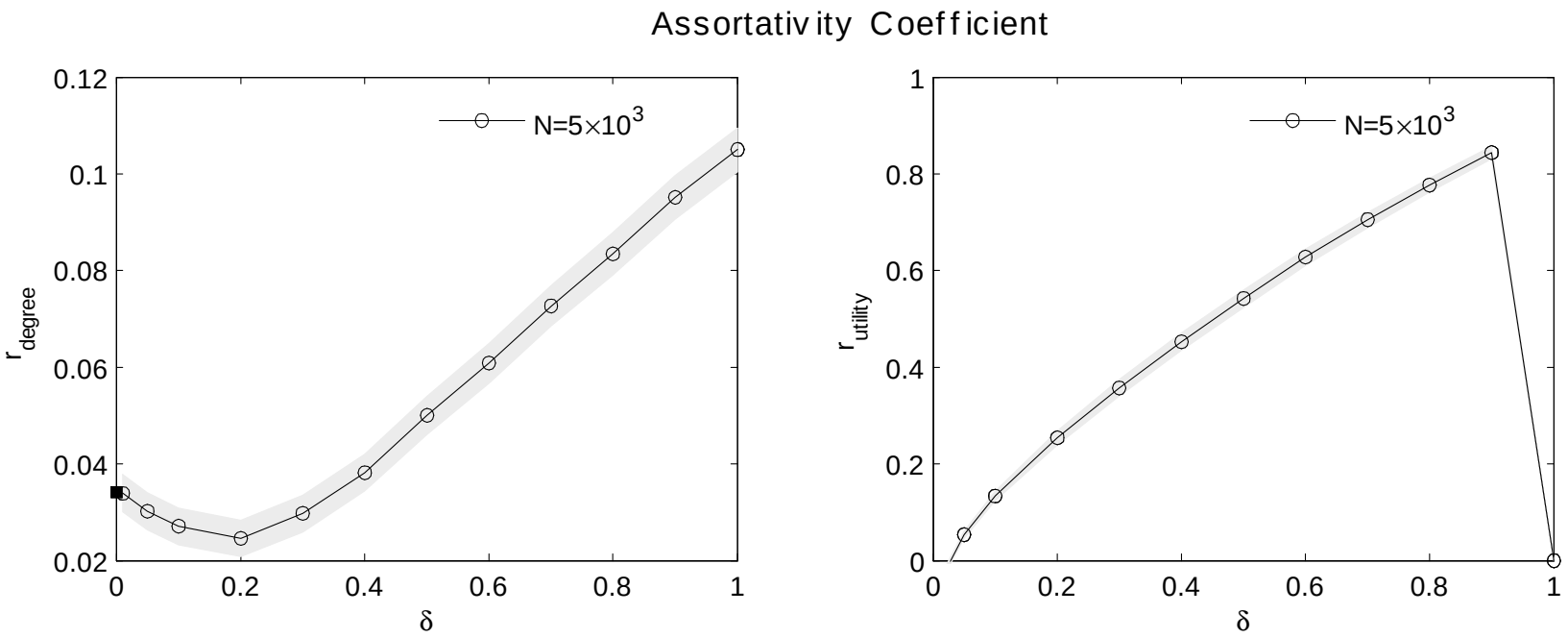
Evolving networks

Rich-get-richer, fit-get-rich and exponential regimes

- $\delta \to 0$ —*Rich-get-richer regime*. Preferential attachment by degree is indistinguishable from preferential attachment by utility (for $\delta = 0$ the degree distribution becomes exponential);
- $\delta \simeq 0.15$ ($N = 10^5$) —*Fit-get-rich regime*.
  - Properties:
    - minimum of average path length and assortativity by degree;
    - stratification of utility values (distribution of utility shows a step-like behaviour);
  - The neighbours of the utility hubs have high utility, but low degree (degree assortativity is minimal), so the network grows from a relatively isolated core of high utility nodes linked to nodes which have low connectivity;
- $\delta = 1$ —*Exponential regime*. The degree distribution becomes exponential.

Evolving Networks

Assortativity



Evolving Networks

Average path length and clustering coefficient

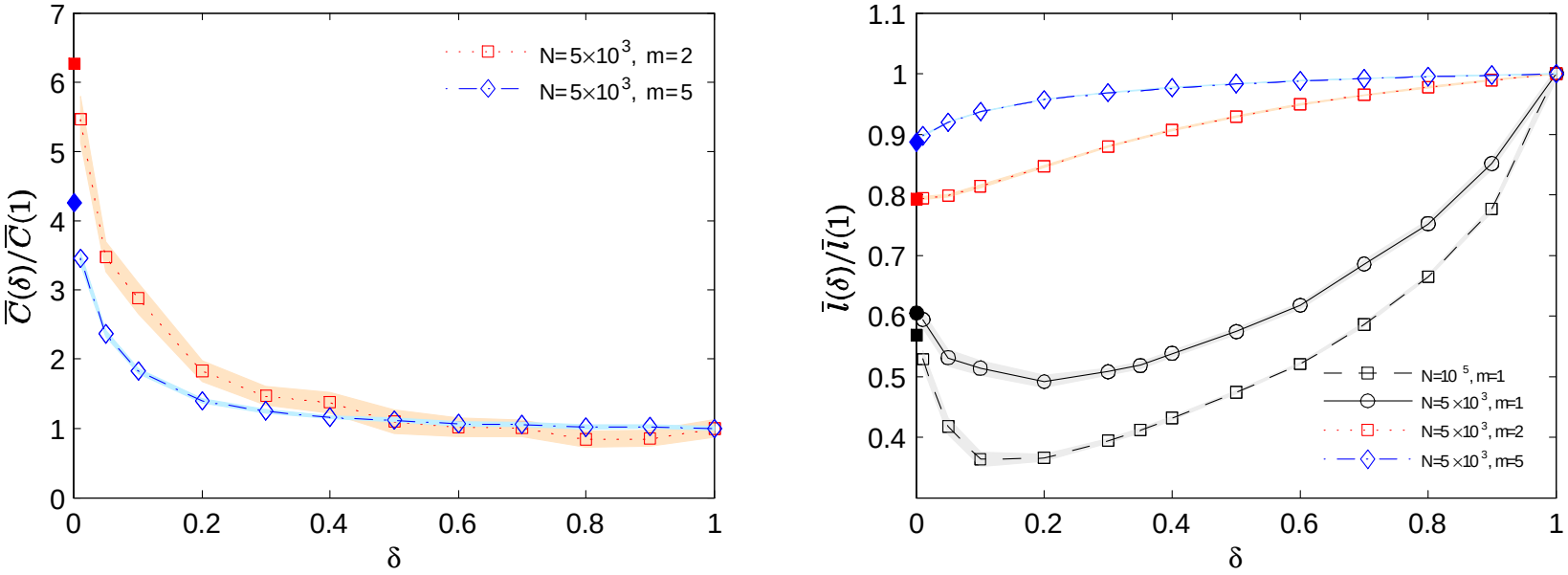
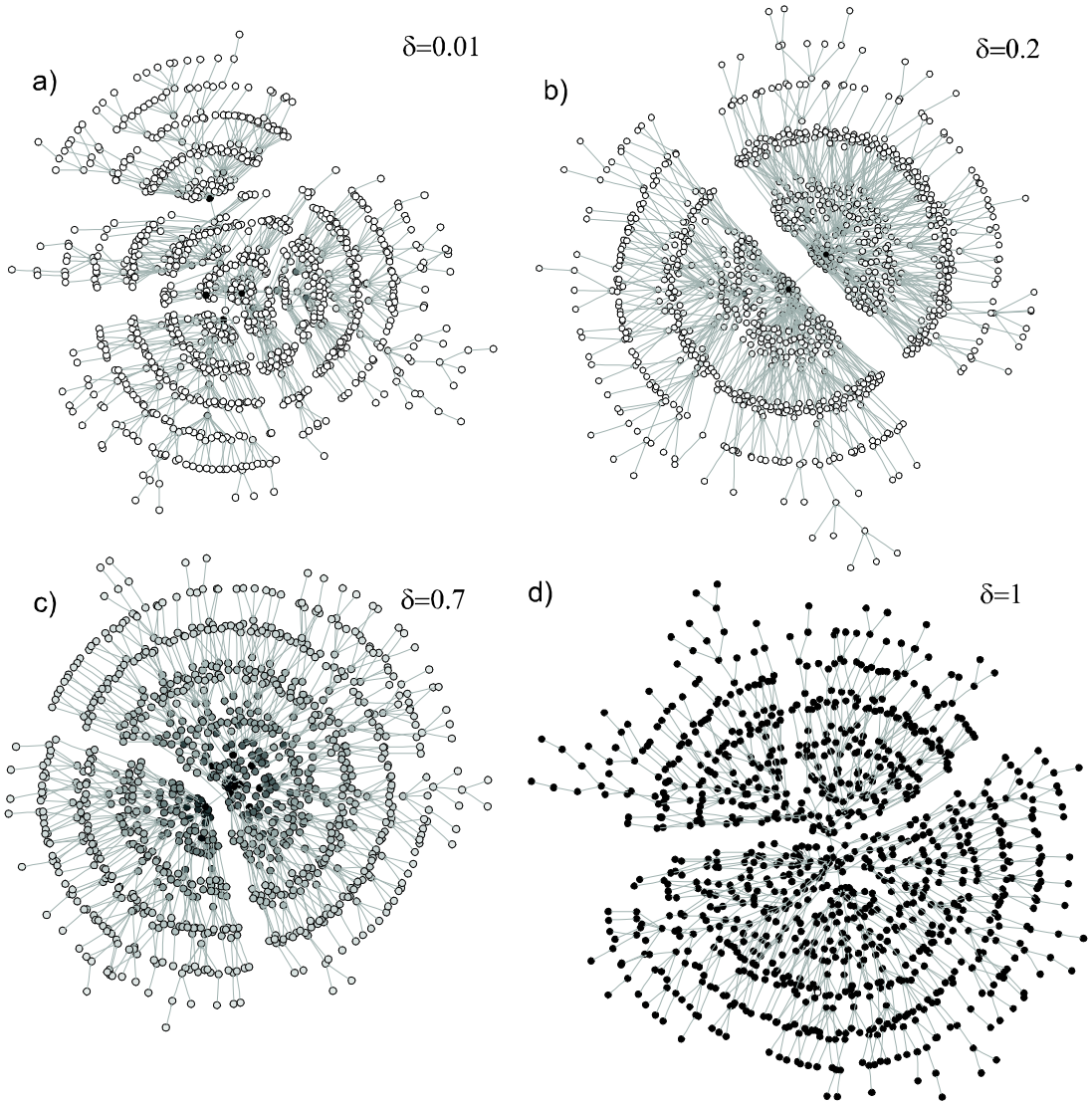


Figure: Average clustering coefficient, a), and path length, b), for the simulation results when $m = 1, 2$ and 5. Curves were scaled by, respectively, average path length and clustering coefficient for $\delta = 1$ (Poisson network). Coloured bands around the curves are 95% confidence intervals.

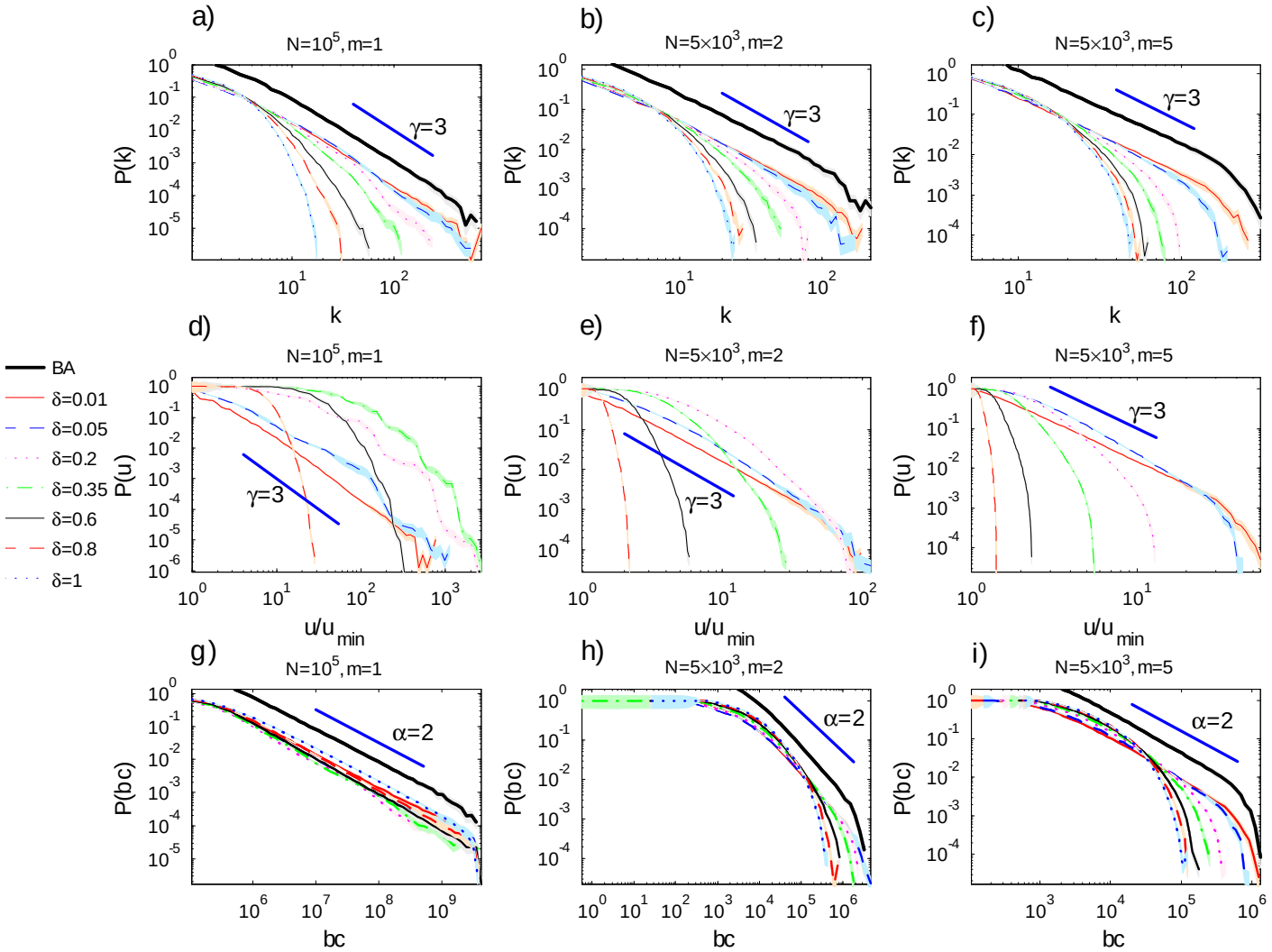
Evolving Networks

Network Layouts



Evolving networks

Cumulative distributions



Discussion

- We have studied models of socioeconomic networks with long-range interactions inspired by the work of Jackson and Wolinsky;
- Average utility in Poisson and scale-free networks:
  - Scale-free networks have higher $\overline{u}$ for the range of parameters that is of significance in real-world networks ($z_1 \geq 2$);
  - When $z_1 = 2$, the star has the highest utility of the networks studied here... but the star is of little practical relevance;
- We have proposed a natural extension of the Barabási–Albert preferential attachment by degree to preferential attachment by utility;
- For small δ, preferential attachment by utility is stronger than by degree: the neighbours of the utility hubs have high utility, but low degree, so the network grows from a "core" of high utility around the utility hubs;
- We have identified three regimes as δ is varied: rich-get-richer ($\delta \to 0$), fit-get-richer (δ small) and exponential ($\delta = 1$);

Thanks

Thank you for your attention.

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